SOME OPEN MAPPING THEOREMS FOR MEASURES

BY

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Abstract. Given a compact Hausdorff space X, let $C(X)$ be the Banach space of continuous real valued functions on X with sup norm and let $M(X)$ be its dual considered as finite regular Borel measures on X. Let $U(X)$ denote the closed unit ball of $M(X)$ and let $P(X)$ denote the nonnegative measures in $M(X)$ of norm 1. A continuous map φ of X onto another compact Hausdorff space Y induces a natural linear transformation π of $M(X)$ onto $M(Y)$ defined by setting $\pi(\mu)(g) = \mu(g \circ \varphi)$ for $\mu \in M(X)$ and $g \in C(Y)$. It is shown that π is norm open on $U(X)$ and on $A \cdot P(X)$ for any subset A of the real numbers. If φ is open, then π is weak* open on $A \cdot P(X)$. Several examples are given which show that generalization in certain directions is not possible. The paper concludes with some remarks about continuous selections.

1. **Introduction.** Let E_1 and E_2 be Banach spaces with closed unit balls U_1 and U_2, respectively, and let π be a linear transformation of E_1 onto E_2 which carries U_1 onto U_2. Of course, π is an open map of E_1 onto E_2, but there are simple examples which illustrate that π is not necessarily open as a map of U_1 onto U_2 (see §5). It can be shown, though, that if U_1 is sufficiently round (precisely, *locally uniformly convex* [1, p. 113]), then π will be open on U_1. However, as indicated by the examples in §5, in the absence of any roundness conditions, general theorems of this kind are hard to come by.

We shall be concerned with open mapping theorems in the following setting. Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of continuous real-valued functions on X with the sup norm. The dual of $C(X)$ is the space $M(X)$ of real regular Borel measures on X of finite total variation. Denote the closed unit ball in $M(X)$ by $U(X)$, the nonnegative (resp. nonpositive) measures in $M(X)$ by $M^+(X)$ (resp. $M^-(X)$), and the measures in $M^+(X)$ of norm 1 by $P(X)$ (the probability measures on X). Given another compact Hausdorff space Y and continuous map φ of X onto Y, there is a natural linear transformation π of $M(X)$ onto $M(Y)$ which takes $\mu \in M(X)$ onto the measure $\pi(\mu)$ defined on the Borel sets B of Y by

$$\pi(\mu)(B) = \mu(\varphi^{-1}(B)),$$

or, equivalently, on the functions $g \in C(Y)$ by

$$\pi(\mu)(g) = \mu(g \circ \varphi).$$

The map π is the dual of the isometric embedding of $C(Y)$ into $C(X)$ which takes $g \in C(Y)$ into $g \circ \varphi$. It follows from the Hahn-Banach theorem, then, that $\pi(U(X)) = U(Y)$. Moreover, since π is a positive operator and $\pi(\mu)(1) = \mu(1)$, we have $\pi(P(X)) = P(Y)$, and, consequently, $\pi(M^+(X)) = M^+(Y)$. In fact, $\pi(A \cdot P(X)) = A \cdot P(Y)$ for any set A of real numbers.

It is shown in §2 that, in either the norm or weak* topologies, π is open on $M^+(X)$ if and only if π is open on $A \cdot P(X)$ for every subset A of R, the real numbers. We then show in §3 that π is norm open on $U(X)$ and on $M^+(X)$. In §4, we show that π is weak* open on $M^+(X)$ (or $P(X)$) if and only if φ is open. One of the examples shows that π is generally not weak* open on $U(X)$, even if φ is open. We conclude with some remarks about selection theorems.

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2. **Some preliminary remarks.** In either the norm or weak* topologies, it is clear that a subset V of $M^+ \cup M^-$ is open in $M^+ \cup M^-$ if and only if $V \cap M^+$ is open in M^+ and $V \cap M^-$ is open in M^-. Hence, π is open on $M^+(X)$ (or $M^-(X)$) if and only if π is open on $M^+(X) \cup M^-(X)$. Moreover, for any subset V of $M^+(X) \cup M^-(X)$ and any $A \subseteq R$,

$$\pi(V \cap A \cdot P(X)) = \pi(V) \cap A \cdot P(Y).$$

Hence, we have the following.

LEMMA 2.1. *In either the norm or weak* topologies, the map π is open on $M^+(X)$ if and only if π is open on $A \cdot P(X)$ for every set A of real numbers.*

For any nonnegative measure μ_0, let $[0, \mu_0]$ denote the set of all measures μ satisfying $0 \leq \mu \leq \mu_0$.

LEMMA 2.2. *For any $\mu_0 \in M^+(X)$, $\pi[0, \mu_0] = [0, \pi(\mu_0)]$.*

Proof. Since π is positive, $\pi[0, \mu_0] \subseteq [0, \pi(\mu_0)]$. Moreover, $[0, \mu_0]$ is a weak* compact convex subset of $M(X)$ and π is weak* continuous. Hence, $\pi[0, \mu_0]$ is a weak* compact convex subset of $M(Y)$. Therefore, if $0 \leq \nu \leq \pi(\mu_0)$ and it is assumed that $\nu \notin \pi[0, \mu_0]$, then, by the separation theorem, there exists $g \in C(Y)$ and $\varepsilon > 0$ such that, for all $\mu \in [0, \mu_0]$,

$$\int g \, d\nu \geq \varepsilon + \int g \circ \varphi \, d\mu.$$

Let $S = \{x \in X : g(\varphi(x)) \geq 0\}$ and let $\omega = \mu_0 | S$. Then $\omega \in [0, \mu_0]$ and

$$\int g \circ \varphi \, d\omega = \int g^+ \, d\pi(\mu_0) \geq \int g \, d\nu,$$

which is a contradiction.

REMARK. Lemma 2.2 is, in fact, a special case of the following result. Let F be a Banach space and vector lattice and let E be a closed vector sublattice. If $\{f^+ : f \in F, \|f\| = 1\}$ is norm bounded, then the restriction map π of F^* onto E^* maps intervals onto intervals. (Given $\mu_0 \geq 0$ in F^* and $\nu \in [0, \pi(\mu_0)]$ in E^*, define the sublinear functional p on F by $p(f) = \mu_0(f^+)$ and apply the Hahn-Banach theorem.)

3. **Open mapping theorems for the norm topology.**   For any subset K of $M(X)$, π is open on K if and only if π is open at every point μ in K, i.e., for every neighborhood V of μ in K, $\pi(V)$ is a neighborhood of $\pi(\mu)$ in $\pi(K)$. For the norm topology, this is equivalent to saying that for every sequence ν_n in $\pi(K)$ which converges in norm to $\pi(\mu)$, there is a sequence μ_n in K which converges in norm to μ and $\pi(\mu_n) = \nu_n$ for all n.

THEOREM 3.1. *π is a norm open mapping of $M^+(X)$ onto $M^+(Y)$.*

Proof. Fix $\mu_0 \in M^+(X)$ and let $\nu_0 = \pi(\mu_0)$. Suppose $\nu_n \in M^+(Y)$ and $\nu_n \to \nu_0$. Since π is open on $M(X)$, there exist $\omega_n \to \mu_0$ and $\pi(\omega_n) = \nu_n$. Then $\pi(\omega_n^+) \geq \nu_n$ and, by norm continuity of the lattice operations, $\omega_n^+ \to \mu_0$. By Lemma 2.2, there exist $\mu_n \in [0, \omega_n^+]$ satisfying $\pi(\mu_n) = \nu_n$. Since $\omega_n^+ - \mu_n \geq 0$ and π is an isometry on $M^+(X)$, we have

$$\|\omega_n^+ - \mu_n\| = \|\omega_n^+\| - \|\mu_n\| = \|\omega_n^+\| - \|\nu_n\|,$$

which converges to $\|\mu_0\| - \|\nu_0\| = 0$. It follows that $\mu_n \to \mu_0$. Hence, on $M^+(X)$, π is open at μ_0.

THEOREM 3.2. *π is a norm open mapping of $U(X)$ onto $U(Y)$.*

Proof. Fix $\mu_0 \in U(X)$ and let $\nu_0 = \pi(\mu_0)$.

Suppose $\|\nu_0\| < 1$ and let V be any open neighborhood of μ_0 in $U(X)$. Then for some ω_0 in $U(X)$, $\pi(\omega_0) = \nu_0$ and $\|\omega_0\| = \|\nu_0\|$, and for some real number t in the open interval $(0, 1)$, $\rho = t\omega_0 + (1-t)\mu_0$ is in V. Since π is open on $M(X)$, $\pi(\rho) = \nu_0$, and $\|\rho\| < 1$, it follows that $\pi(V)$ is a neighborhood of ν_0.

Now assume $\|\nu_0\| = 1$. Then $\|\mu_0\| = 1$. Suppose $\nu_n \in U(Y)$ and $\nu_n \to \nu_0$. Since π is open on $M(X)$, there exists ω_n in $M(X)$ such that $\omega_n \to \mu_0$ and $\pi(\omega_n) = \nu_n$. We modify the ω_n's to bring them into $U(X)$. Since $\nu_n = \pi(\omega_n^+) - \pi(\omega_n^-)$, it follows that $\pi(\omega_n^+) \geq \nu_n^+$ and $\pi(\omega_n^-) \geq \nu_n^-$. By Lemma 2.2, there exists $\rho_n \in [0, \omega_n^+]$ and $\lambda_n \in [0, \omega_n^-]$ satisfying $\pi(\rho_n) = \nu_n^+$ and $\pi(\lambda_n) = \nu_n^-$. Set $\mu_n = \rho_n - \lambda_n$. Then $\pi(\mu_n) = \nu_n$ and $\mu_n \in U(X)$ since

$$\|\mu_n\| \leq \|\rho_n\| + \|\lambda_n\| = \|\nu_n^+\| + \|\nu_n^-\| = \|\nu_n\| \leq 1.$$

Moreover, since $0 \leq \rho_n \leq \omega_n^+$ and $0 \leq \lambda_n \leq \omega_n^-$,

$$\|\omega_n - \mu_n\| \leq \|\omega_n^+ - \rho_n\| + \|\omega_n^- - \lambda_n\| \leq \|\omega_n^+\| - \|\rho_n\| + \|\omega_n^-\| - \|\lambda_n\| = \|\omega_n\| - \|\nu_n\|.$$

Hence, $\mu_n \to \mu_0$ and, on $U(X)$, π is open at μ_0.

4. **Open mapping theorems for the weak* topology.**   In this section, we prove the equivalence of the three statements: (i) π is weak* open on $P(X)$, (ii) φ is open, and (iii) π is weak* open on $M^+(X)$. By Lemma 2.1, it suffices to prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

PROPOSITION 4.1. *If π is a weak* open map of $P(X)$ onto $P(Y)$, then φ is open.*

Proof. Consider any $x_0 \in X$ and let $y_0 = \varphi(x_0)$. Let $f \in C(X)$ satisfy $f(x_0) = 0$ and $f \geqq 0$. Set

$$G = \{x \in X : f(x) < 1\} \quad \text{and} \quad V = \{\mu \in P(X) : \mu(f) < 1\}.$$

Since the sets G obtained in this manner form a neighborhood base at x_0, it suffices to show that $\varphi(G)$ is a neighborhood of y_0. But $\pi(V)$ is a weak* neighborhood of δ_{y_0}, the unit point mass at y_0. Since for any $\nu \in P(Y)$ and $g \in C(Y)$,

$$|\nu(g) - g(y_0)| \leqq \nu(|g - g(y_0)|),$$

it follows that there exist finitely many functions $g_1, \ldots, g_n$ in $C(Y)$ such that $g_i(y_0) = 0$, $g_i \geqq 0$, for $i = 1, \ldots, n$, and $W \subset \pi(V)$, where

$$W = \{\nu \in P(Y) : \nu(g_i) < 1, i = 1, \ldots, n\}.$$

Set

$$H = \{y \in Y : g_i(y) < 1, i = 1, \ldots, n\}.$$

Then H is a neighborhood of y_0 and $W \subset \pi(V)$ implies that $H \subset \varphi(G)$. For, if $y \in H$, then $\delta_y \in W$, so that $\delta_y = \pi(\mu)$ for some $\mu \in V$. Hence, μ is supported on $\varphi^{-1}(y)$ and $\mu(f) < 1$, which implies that $f(x) < 1$ for some $x \in \varphi^{-1}(y)$.

In the proofs given below, we require the notion of set convergence. The definitions and basic facts required are as follows (see [2]). Let K be a topological space and let $F(K)$ denote the closed nonempty subsets of K. Given finitely many open sets $V_1, \ldots, V_n$ in K, let $\langle V_1, \ldots, V_n \rangle$ denote the collection of sets A in $F(K)$ satisfying $A \subset \bigcup_{i=1}^{n} V_i$ and $A \cap V_i \neq \varnothing$ for $i = 1, \ldots, n$. The sets of the form $\langle V_1, \ldots, V_n \rangle$ constitute a base for a topology on $F(K)$, called the *finite topology*. If K is compact Hausdorff, then $F(K)$ is compact Hausdorff, and if $A_\alpha \to A$ in $F(K)$, for some net A_α, then A is the set consisting of those points $a \in K$ for which $V \cap A_\alpha \neq \varnothing$ eventually, for every neighborhood V of a.

LEMMA 4.2. *If K is a compact subset of a linear topological space and A_α is a net of closed convex subsets of K which converge to a set A in the finite topology, then A is convex.*

Proof. If $\mu, \nu \in A$ and $0 \leqq t \leqq 1$, then any neighborhood W of $t\mu + (1 - t)\nu$ contains $tU + (1 - t)V$ for some neighborhoods U, V of μ, ν respectively. Since $U \cap A_\alpha$ and $V \cap A_\alpha$ are eventually nonempty and each A_α is convex, it follows that $W \cap A_\alpha$ is eventually nonempty. Hence, $t\mu + (1 - t)\nu \in A$.

LEMMA 4.3. *Suppose φ is an open map of X onto Y. Then for each $f \in C(X)$, the real-valued function f^* defined on Y by*

$$f^*(y) = \sup \{f(x) : \varphi(x) = y\}$$

is continuous.

Proof. Let r be any real number. Then

$$\{y \in Y : f^*(y) > r\} = \varphi(\{x \in X : f(x) > r\})$$

is open. Hence, it suffices to show that $\{y \in Y : f^*(y) \geqq r\}$ is closed. Suppose $y_\alpha \to y$ and $f^*(y_\alpha) \geqq r$ for some net y_α in Y. Choose $x_\alpha \in X$ so that $\varphi(x_\alpha) = y_\alpha$ and $f(x_\alpha) = f^*(y_\alpha)$. If x is any limit point of the net x_α, then $\varphi(x) = y$ and $f(x) \geqq r$, so that $f^*(y) \geqq r$.

THEOREM 4.4. *If φ is an open map of X onto Y, then π is a weak* open map of $M^+(X)$ onto $M^+(Y)$.*

Proof. Fix $\mu_0 \in M^+(X)$ and let $\nu_0 = \pi(\mu_0)$. For $\mu \in M^+(X)$, $\mu(1) = \|\mu\|$, so that for any $r > \|\mu_0\|$, any weak* neighborhood of μ_0 in $M^+(X)$ contains one which is contained in $rU(X)$. Hence, for a fixed $r > \|\mu_0\|$, it suffices to show that if W is a weak* compact neighborhood of μ_0 in $M^+(X)$ and $W \subset K$, where $K = rU(X) \cap M^+(X)$, then $\pi(W)$ is a weak* neighborhood of ν_0 in $M^+(Y)$.

Assume $\pi(W)$ is not a weak* neighborhood of ν_0 in $M^+(Y)$. Since π is weak* continuous, $\pi(W)$ is weak* closed, and since the collection of positive combinations of point masses is weak* dense in $M^+(Y)$, there is a net ν_α in $rU(Y) \cap M^+(Y)$ which converges weak* to ν_0 and is such that, for all α, $\nu_\alpha \notin \pi(W)$ and ν_α is a positive combination of point masses in Y. Let $A_\alpha = \pi^{-1}(\nu_\alpha) \cap K$. Then, by our remarks above, there is a nonempty weak* compact convex set A in K and a subnet A_β of A_α such that $A_\beta \to A$ in the finite topology. Since no A_β meets the interior of W, $\mu_0 \notin A$. Hence, by the separation theorem, there is an $f \in C(X)$ and $\varepsilon > 0$ such that

$$\mu_0(f) \geqq \varepsilon + \mu(f), \quad \text{for all } \mu \in A.$$

Define f^* as in Lemma 4.3. Then $f^* \in C(Y)$. Since, for each β, ν_β is a positive combination of point masses concentrated at points $y_{\beta,k}$ in Y, we may choose corresponding point masses at points $x_{\beta,k}$ in $\varphi^{-1}(y_{\beta,k})$ for which $f(x_{\beta,k}) = f^*(y_{\beta,k})$, and so construct, for each β, a $\mu_\beta \in M^+(X)$ satisfying $\pi(\mu_\beta) = \nu_\beta$ and $\mu_\beta(f) = \nu_\beta(f^*)$. Then $\mu_\beta(f) \to \nu_0(f^*)$, and, if μ is any limit point of the net μ_β, then $\mu \in A$, since $\mu_\beta \in A_\beta$ for each β. Moreover,

$$\mu(f) = \nu_0(f^*) = \mu_0(f^* \circ \varphi) \geqq \mu_0(f),$$

which is a contradiction.

5. **Examples.** In this section we give some examples which help indicate to what extent generalization of the above theorems is not possible.

EXAMPLE 1. Let $X = I \times \{0, 1\}$ and $Y = I$, where I is the closed unit interval $[0, 1]$. Let $\varphi\colon X \to Y$ be the projection map, $\varphi(t, i) = t$. Then φ is open but the induced map π is not weak* open as a map of $U(X)$ onto $U(Y)$. For, if δ_x denotes the unit point mass at x, then $\mu_0 = \frac{1}{2}(\delta_{(1,1)} - \delta_{(1,0)})$ is in $U(X)$ and $\pi(\mu_0) = 0$. Also, $\nu_n = \frac{1}{2}(\delta_0 - \delta_{1/n})$ converges weak* to 0 in $U(Y)$. But, if $\pi(\mu_n) = \nu_n$ and $\mu_n \in U(X)$, then μ_n must be supported on $\varphi^{-1}(\{0, 1/n\})$ and, therefore, cannot converge to μ_0.

EXAMPLE 2. Let E_1 be R^3 with the norm $\|(x, y, z)\|' = (x^2 + y^2)^{1/2} + |z|$ and let E_2 be R^2 with the usual norm $\|(x, y)\| = (x^2 + y^2)^{1/2}$. Denote their closed unit balls by U_1 and U_2 respectively. Define $\pi\colon E_1 \to E_2$ by $\pi(x, y, z) = (x, y+z)$. Then π is a linear transformation which maps U_1 onto U_2. If $z \neq 0$ and $\|(x, y, z)\|' = \|(x, y+z)\| = 1$, then $x = 0$ and $|y+z| = 1$. Therefore, $\{(x, y, z) : z > 0\} \cap U_1$ is a neighborhood of $(0, 0, 1)$ in U_1 whose image under π contains no points of norm 1 other than $(0, 1)$. Hence, on U_1, π is not open at $(0, 0, 1)$.

EXAMPLE 3. Let m be Lebesgue measure on $[0, 1]$ and let $\mu = m + \delta_0$, where δ_0 is the unit point mass at 0. Define $\pi\colon L_1(\mu) \to L_1(m)$ by $\pi(f) = f + f(0)$. Then π is a well-defined linear transformation which maps the closed unit ball U of $L_1(\mu)$ onto the closed unit ball U' of $L_1(m)$. Moreover, π is positive and is an isometry on positive functions. However, π does not preserve intervals nor is π open on U. We see this as follows. Let χ_A denote the characteristic function of the set A and χ_0 the characteristic function of $\{0\}$. Then $\pi(\chi_0) = 1$ and, if $f \in L_1(\mu)$ and $0 \leq f \leq \chi_0$, then $f = c\chi_0$ for $0 \leq c \leq 1$. Hence, $\pi[0, \chi_0] \neq [0, \pi(\chi_0)]$. Also, on U, π is not open at χ_0. For $\chi_{[1/n,1]} \to 1$ in $L_1(m)$ and, if it is assumed there exist $f_n \in U$ such that $\pi(f_n) = \chi_{[1/n,1]}$ and $f_n \to \chi_0$ in $L_1(\mu)$, then $f_n(0) \to 1$ and, in $L_1(m)$,

$$f_n = -f_n(0)\chi_{[0,1/n)} + (1 - f_n(0))\chi_{[1/n,1]},$$

so that we eventually have

$$\int |f_n|\, d\mu = f_n(0)/n + (1 - f_n(0))(1 - 1/n) + f_n(0)$$
$$= 1 + (2/n)(f_n(0) - \tfrac{1}{2}),$$

contradicting the assumption that every $f_n \in U$.

EXAMPLE 4. Let $X = I \times \{0, 1\}$ and $Y = I$ where $I = [0, 1]$. Define a linear transformation π from $C(X)$ to $C(Y)$ by $\pi(f)(t) = (1-t)f(t, 0) + tf(t, 1)$. Then π is a positive operator which maps the closed unit ball U of $C(X)$ onto the closed unit ball U' of $C(Y)$. We show that π is not open as a map from U to U'. Let $f_0 \in U$ be defined by $f_0(t, i) = i$ for $(t, i) \in X$. Then $\pi(f_0)(t) = t$ for all $t \in I$. Set $g_n(t) = \min(1, t + 1/n)$. Then $g_n \in U'$ and $g_n \to \pi(f_0)$ in $C(Y)$. Suppose that $f_n \in U$ and $\pi(f_n) = g_n$. Then, for $1 - 1/n \leq t \leq 1$, $g_n(t) = 1$, so that $f_n(t, 0) = f_n(t, 1) = 1$ and, hence, $\|f_0 - f_n\| \geq 1$. Therefore, on U, π is not open at f_0.

6. **Selection theorems.** By the selection theorems of Michael [3], [4], each of the above open mapping theorems gives rise to a corresponding theorem about continuous selections.

Let $K(X)$ denote either $U(X)$ or $A \cdot P(X)$, for some fixed set A of real numbers, with the norm topology (or $A \cdot P(X)$ with the weak* topology, if X is metrizable and φ is open), and let $K(Y)$ be its counterpart in $M(Y)$.

COROLLARY 6.1. *Let S be a paracompact space, S_0 a closed subset of S, and $\theta: S \to K(Y)$ a continuous map. Let $\theta_0 = \theta|S_0$, the restriction of θ to S_0. Then, for any continuous map $\theta_0': S_0 \to K(X)$ satisfying $\pi\theta_0' = \theta_0$, there is a continuous map $\theta': S \to K(X)$ satisfying $\pi\theta' = \theta$ and $\theta'|S_0 = \theta_0'$.*

Proof. Since π is open on $K(X)$ and θ is continuous, it follows that the map from S to the nonempty closed convex subsets of $K(X)$ which takes $s \in S$ into $\pi^{-1}(\theta(s)) \cap K(X)$ is lower semicontinuous. The map θ_0' is a continuous selection for this mapping on S_0. In the norm case, a continuous selection θ' extending θ_0' to all of S is guaranteed by [3, Proposition 1.4] and [4, Theorem 1.2] (see also [5, Theorem 1]). In the case of the weak* topology, in $M^+(X) \cup M^-(X)$ convergence of a net μ_α at the constant function 1 implies convergence of the norms $\|\mu_\alpha\|$, so that pointwise convergence on a dense subset of $C(X)$ containing 1 is equivalent to weak* convergence. Since $C(X)$ is separable when X is metrizable, it follows that the weak* topology of $M^+(X) \cup M^-(X)$ is metrizable. Moreover, the sets $\pi^{-1}(\theta(s)) \cap K(X)$ are weak* compact. Therefore, in the weak* case, when X is metrizable and φ is open [4, Theorem 1.2], still applies.

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